

GUNNING-NARASIMHAN'S THEOREM WITH A GROWTH CONDITION

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ABSTRACT. Given a compact Riemann surface X and a point $x_0 \in X$, we construct a holomorphic function without critical points on the punctured Riemann surface $R = X \setminus \{x_0\}$ which is of finite order at x_0 .

1. THE STATEMENT

Let X be a compact Riemann surface, let x_0 be an arbitrary point of X , and let $R = X \setminus \{x_0\}$. The set of holomorphic functions on R will be denoted by $\mathcal{O}(R)$. Let $U \subset X$ be a coordinate neighborhood of the point x_0 and let z be a local coordinate on U with $z(x_0) = 0$. A holomorphic function $f \in \mathcal{O}(R)$ on R is said to be of *finite order* (at the point x_0) if there exist positive numbers λ and μ such that

$$(1) \quad |f(z)| \leq \lambda \exp |z|^{-\mu} \quad \text{holds on } U \setminus \{x_0\}.$$

We denote by $\mathcal{O}_{f.o.}(R)$ the set of all holomorphic functions of finite order on R . For any $f \in \mathcal{O}_{f.o.}(R)$, the *order* of f is defined as the infimum of all numbers $\mu > 0$ such that (1) holds for some $\lambda > 0$. By using Poisson-Jensen's formula it is easy to see that, for any nonvanishing holomorphic function f on $U \setminus \{x_0\}$ satisfying (1), there exist a neighborhood $V \ni x_0$ and a number $\chi > 0$ such that $\frac{1}{|f(z)|} \leq \chi \exp |z|^{-\mu}$ on $V \setminus \{x_0\}$ (Hadamard's theorem, c.f. [A, Chap. 5]).

In 1967 Gunning and Narasimhan proved that every open Riemann surface admits a holomorphic function without critical points [GN]. Our goal is to prove the following result for punctured Riemann surfaces.

Theorem 1.1. *If X is a compact Riemann surface and $x_0 \in X$ then the punctured Riemann surface $R = X \setminus \{x_0\}$ admits a noncritical holomorphic function of finite order; that is, $\{f \in \mathcal{O}_{f.o.}(R) : df \neq 0 \text{ everywhere}\} \neq \emptyset$.*

We show that this result is the best possible one, except when $X = \mathbb{CP}^1$ is the Riemann sphere in which case $R = \mathbb{C}$:

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Proposition 1.2. *If X is a compact Riemann surface of genus $g \geq 1$ and $x_0 \in X$ then every algebraic function $X \setminus \{x_0\} \rightarrow \mathbb{C}$ has a critical point.*

In the case when X is a torus, this was shown in [M, §4].

Proof. Assume that $f: R = X \setminus \{x_0\} \rightarrow \mathbb{C}$ is an algebraic function. Then f extends to a meromorphic map $X \rightarrow \mathbb{CP}^1$ sending x_0 to the point ∞ . Let d denote the degree of f at x_0 , so f equals the map $z \mapsto z^d$ in a certain pair of local holomorphic coordinates at the points x_0 and ∞ . Since $f^{-1}(\infty) = \{x_0\}$, d is also the global degree of f . By the Riemann-Hurwitz formula (see [Ha]) we then have

$$\chi(X) = d\chi(\mathbb{CP}^1) - b,$$

where $\chi(X)$ is the Euler number of X and b is the total branching order of f (the sum of its local branching orders over the points of X). If we assume that f has no critical points on R , then it only branches at x_0 , and its branching order at x_0 is clearly $b = d - 1$. Hence the above equation reads $2 - 2g = 2d - (d - 1) = d + 1 \geq 1$ which is clearly impossible if $g \geq 1$. In fact, we see that any algebraic function $f: R = X \setminus \{x_0\} \rightarrow \mathbb{C}$ with degree d at x_0 must have precisely $(d + 1) - (2 - 2g) = d + 2g - 1$ branch points in R when counted with algebraic multiplicities. \square

2. PRELIMINARIES

We assume that X and $R = X \setminus \{x_0\}$ are as above.

Proposition 2.1. *For any effective divisor δ on X whose support does not contain the point x_0 there exists $f \in \mathcal{O}_{f.o.}(R)$ whose zero divisor $f^{-1}(0)$ coincides with δ .*

Proof. Since holomorphic vector bundles over noncompact Riemann surfaces are trivial by Grauert's Oka principle, there exists a holomorphic function f_0 on R whose zero divisor equals δ . Let V be a disc neighborhood of the point x_0 in X , with a holomorphic coordinate z in which $z(x_0) = 0$, such that f_0 does not vanish on $V \setminus \{x_0\}$. Let $m \in \mathbb{Z}$ denote the winding number of f around the point x_0 . Choose a meromorphic function h on X such that $h(z) = c(z)z^m$ on $z \in V$ for some nonvanishing holomorphic function c on V , and such that all remaining zeros and poles of h lie in $X \setminus \overline{V}$. Then f_0/h is a nowhere vanishing holomorphic function with winding number zero in $V \setminus \{x_0\}$, and hence $\log(f_0/h)$ has a single valued holomorphic branch on $V \setminus \{x_0\}$. Choose a smaller disc $W \Subset V$ centered at x_0 . By solving a Cousin-I problem we find a holomorphic functions u_1 on $X \setminus \overline{W}$ and u_2 on $V \setminus \{x_0\}$ such that $u_1 - u_2 = \log(f_0/h)$ holds on $V \setminus \overline{W}$, and such that x_0 is a pole of the function u_2 . Hence, letting $f = he^{-u_2}$ on $V \setminus \{x_0\}$ and $f = f_0e^{-u_1}$ on $R \setminus W$, we obtain a function $f \in \mathcal{O}_{f.o.}(R)$ satisfying $f^{-1}(0) = \delta$. \square

Let $L \rightarrow X$ be a holomorphic line bundle and let h be a fiber metric of L . A holomorphic section s of L over R is said to be of finite order if the length $|s|$ of s with respect to h satisfies on $U \setminus \{x_0\}$, as a function of the local coordinate z , that

$$|s|(z) \leq \lambda \exp |z|^{-\mu} \quad \text{for some } \lambda, \mu \in (0, \infty).$$

The order of s is defined similarly as in the case of holomorphic functions. Since every holomorphic line bundle over X is associated with a divisor, Proposition 2.1 implies the following:

Proposition 2.2. *For any holomorphic line bundle L over X , there exists a holomorphic section s of the restricted bundle $L|_R$ such that s is of finite order and $s(x) \neq 0$ for all $x \in R$.*

Proof. Let v be any meromorphic nonzero section of L . Let p_1, \dots, p_m (resp. q_1, \dots, q_n) be the poles (resp. the zeros) of v in R . By Proposition 2.1 there exist functions $f, g \in \mathcal{O}_{f.o.}(R)$ such that $p_1 + p_2 + \dots + p_m$ (resp. $q_1 + \dots + q_n$) is the zero divisor of f (resp. of g). Then the section $s = fv/g$ satisfies the stated properties. \square

Corollary 2.3. *There exists a holomorphic 1-form of finite order on R which does not vanish anywhere.*

Let ω be a nowhere vanishing holomorphic 1-form of finite order on R guaranteed by Corollary 2.3. Then, Theorem 1.1 is equivalent to saying that there exists a function $g \in \mathcal{O}_{f.o.}(R)$ such that $g^{-1}(0) = \emptyset$ and $\int_\gamma g\omega = 0$ holds for any 1-cycle γ on R , for the primitives of $g\omega$ will then be without critical points and clearly of finite order, the converse being obvious.

We shall show that such g can be found in a subset of $\mathcal{O}_{f.o.}(R)$ consisting of functions of the form $\exp \int_{x_1}^x \eta$ where η are meromorphic 1-forms on X which are holomorphic on R and $x_1 \in R$ is an arbitrary fixed point in R .

Let us denote by $\Omega_{alg}^1(R)$ (resp. $\mathcal{O}_{alg}(R)$) the set of meromorphic 1-forms (resp. meromorphic functions) on X which are holomorphic on R . The general theory of coherent algebraic sheaves on affine algebraic varieties implies the following (c.f. [S] or [Ha]).

Proposition 2.4. *Every element of $H^1(R; \mathbb{C})$ is represented by an element of $\Omega_{alg}^1(R)$ as a de Rham cohomology class.*

Let K be a compact set in R and let $\mathcal{O}(K)$ denote the set of all continuous functions on K which are holomorphically extendible to some open neighborhoods of K in X . Then the Runge approximation theorem says the following in our situation.

Proposition 2.5. *For any compact set $K \subset R$ such that $R \setminus K$ is connected, the image of the restriction map $\mathcal{O}_{alg}(R) \rightarrow \mathcal{O}(K)$ is dense with respect to the topology of uniform convergence.*

The proof of Theorem 1.1 to be given below is basically a combination of Corollary 2.3, Proposition 2.4 and Proposition 2.5. In order to make a short cut argument, we shall apply a refined version of Proposition 2.5 (Mergelyan's theorem) below.

3. PROOF OF THEOREM 1.1

For any \mathcal{C}^1 curve $\alpha: [0, 1] \rightarrow X$ we denote by $|\alpha|$ its trace, i.e., $|\alpha| = \{\alpha(t): 0 \leq t \leq 1\}$. If α is closed ($\alpha(0) = \alpha(1)$), we denote by $[\alpha]$ its homology class in $H_1(X; \mathbb{Z})$.

Let g denote the genus of X . There exist simple closed real-analytic curves $\alpha_1, \dots, \alpha_{2g}$ in R satisfying

$$(2) \quad H_1(X; \mathbb{Z}) = \sum_{i=1}^{2g} \mathbb{Z}[\alpha_i]$$

such that $\cap_{i=1}^{2g} |\alpha_i| = \{p\}$ holds for some point $p \in R$ and such that, putting $\Gamma = \cup_{i=1}^{2g} |\alpha_i|$, the complement $R \setminus \Gamma$ is connected.

Let ω nowhere vanishing holomorphic 1-form of finite order on R furnished by Corollary 2.3. For each curve α_i there is a neighborhood $U_i \supset |\alpha_i|$ in R and a biholomorphic map φ_i from an annulus $A_r = \{w \in \mathbb{C}: 1-r < |w| < 1+r\}$ onto U_i for a sufficiently small $r > 0$ such that the positively oriented unit circle $\{|w| = 1\}$ is mapped by φ_i onto the curve α_i , with $\varphi_i(1) = p$. For $i = 1, \dots, 2g$ put

$$\phi_i^* \omega = H_i(w) dw, \quad n_i = \frac{1}{2\pi\sqrt{-1}} \int_{|w|=1} d \log H_i \in \mathbb{Z}.$$

By Proposition 2.4 there exists $\xi \in \Omega_{alg}^1(R)$ such that

$$(3) \quad n_i = \frac{1}{2\pi\sqrt{-1}} \int_{\alpha_i} \xi, \quad i = 1, \dots, 2g.$$

Let

$$u(x) = \exp \int_p^x \xi, \quad x \in R.$$

By (2) and (3) the integral is independent of the path in R . (Note that the cycle around the deleted point x_0 is homologous to zero in R .) Hence the function u is well defined, single-valued and nonvanishing on R , and $u \in \mathcal{O}_{f.o.}(R)$ because $\xi \in \Omega_{alg}^1(R)$. Replacing ω by ω/u we obtain a nowhere vanishing 1-form of finite order on R , still denoted ω , for which the winding numbers n_i in (3) equal zero. It follows that for every $i = 1, \dots, 2g$ we have

$$\phi_i^* \omega = e^{h_i(w) + c_i} dw$$

for some constants $c_i \in \mathbb{C}$ and holomorphic function h_i on the annulus $A_r \subset \mathbb{C}$ with $h_i(1) = 0$. Note that the functions $h_i \circ \varphi_i^{-1}: |\alpha_i| \rightarrow \mathbb{C}$ agree at

the unique intersection point p of the curves $|\alpha_i|$, and hence they define a continuous function H on $\Gamma = \cup_i |\alpha_i|$. For every $h \in \mathcal{O}_{alg}(R)$ we have

$$\int_{\alpha_i} e^{-h} \omega = e^{c_i} \int_{|w|=1} e^{h_i - h \circ \varphi_i} dw.$$

These numbers can be made arbitrarily small by choosing h to approximate H uniformly on Γ (which is equivalent to asking that $h_i - h \circ \varphi_i$ is small on $\{|w|=1\}$ for every $i = 1, \dots, 2g$). Such h exist by Mergelyan's theorem: Since $R \setminus \Gamma$ is connected, every continuous function on Γ is a uniform limit of functions in $\mathcal{O}_{alg}(R)$ (c.f. [G, Chap. 3]).

We assert that there exist functions $f_i \in \mathcal{O}_{alg}(R)$ for $i = 1, \dots, 2g$ and a number $\epsilon > 0$ such that, for any $h \in \mathcal{O}_{alg}(R)$ satisfying

$$(4) \quad \sup_{|\alpha_i|} |h_i \circ \varphi_i^{-1} - h| < \epsilon, \quad i = 1, \dots, 2g,$$

there exist numbers $\zeta_i \in \mathbb{C}$ ($i = 1, \dots, 2g$) such that

$$(5) \quad \int_{\alpha_j} \exp \left(\sum_{i=1}^{2g} \zeta_i f_i - h \right) \omega = 0, \quad j = 1, \dots, 2g.$$

To prove this assertion, which clearly implies Theorem 1.1 (the potential of the 1-form under the integral in (5) is a holomorphic function of finite order and without critical points on R), choose functions $f_i \in \mathcal{O}_{alg}(R)$ for $i = 1, \dots, 2g$ satisfying

$$(6) \quad e^{c_j} \int_{|w|=1} f_i \circ \varphi_j(w) dw = \delta_{ij},$$

where δ_{ij} denotes the Kronecker's delta. Such f_i exist by Proposition 2.5 applied with $K = \Gamma$. After fixing the f_i 's, let us choose numbers $0 < \epsilon_0 < 1$ and $C_0 > 1$ in such a way that

$$(7) \quad \sup_{\Gamma} \left| \exp \left(\sum_{i=1}^{2g} \tau_i f_i \right) - 1 - \sum_{i=1}^{2g} \tau_i f_i \right| \leq C_0 \max_i |\tau_i|^2$$

holds if $\tau_i \in \mathbb{C}$ and $\max_i |\tau_i| \leq \epsilon_0$.

Let $c = \max_i |c_i|$. By decreasing the number $\epsilon_0 > 0$ if necessary we can assume that

$$8\pi C_0 e^{1+c} \epsilon_0 < 1.$$

Choose a constant $C_1 > 0$ such that

$$|e^t - 1| < C_1 |t| \quad \text{if } |t| < \epsilon_0.$$

Then, by (6) and (7), it is easy to see that, for any positive number $\epsilon > 0$ satisfying

$$8\pi C_1 \left(1 + \sup_{\Gamma} \sum_{i=1}^{2g} |f_i| \right) \epsilon < \epsilon_0$$

and for any $h \in \mathcal{O}_{alg}(R)$ satisfying (4), the inequality

$$\left| \tau_j - \int_{\alpha_j} \exp \left(\sum \tau_i f_i - h \right) \omega \right| \leq \frac{\epsilon_0}{2}$$

holds for every $j = 1, \dots, 2g$ whenever $\max_i |\tau_i| \leq \epsilon_0$. Hence, for such a choice of h , the map

$$\mathbb{C}^{2g} \ni \tau = (\tau_1, \dots, \tau_{2g}) \xrightarrow{\Phi} (\Phi_1(\tau), \dots, \Phi_{2g}(\tau)) \in \mathbb{C}^{2g},$$

whose j -th component is defined by

$$\Phi_j(\tau) = \int_{\alpha_j} \exp \left(\sum_{i=1}^{2g} \tau_i f_i - h \right) \omega,$$

maps the polydisc $P = \{\tau \in \mathbb{C}^{2g} : \max |\tau_i| < \epsilon_0\}$ onto a neighborhood of the origin in \mathbb{C}^{2g} . In particular, we have $\Phi(\zeta) = 0$ for some point $\zeta = (\zeta_1, \dots, \zeta_{2g}) \in P$, and for this ζ the equations (5) hold. This concludes the proof of Theorem 1.1.

4. CONCLUDING REMARKS

By a minor adjustment of the proof of Theorem 1.1 one can construct a nowhere vanishing holomorphic 1-form of finite order, ω , on R whose periods $\int_{\alpha_j} \omega$ over the basis curves $[\alpha_j]$ of $H_1(R; \mathbb{Z})$ are arbitrary given complex numbers. In other words, one can prove the following result. (See Kusunoki and Sainouchi [KS] and Majcen [M] for the corresponding result on open Riemann surface and without the finite order condition.)

Theorem 4.1. *Let X be a compact Riemann surface and $x_0 \in X$. Every element of the de Rham cohomology group $H^1(X; \mathbb{C})$ is represented by a nowhere vanishing holomorphic 1-form of finite order on $R = X \setminus \{x_0\}$.*

Since every affine algebraic curve $A \subset \mathbb{C}^N$ is obtained by deleting finitely many points from a compact Riemann surface, Theorem 1.1 implies that *every affine algebraic curve admits a noncritical holomorphic function of finite order*. One may ask whether the same result also holds on higher dimensional algebraic manifolds:

Problem 4.2. Does every affine algebraic manifold $A \subset \mathbb{C}^N$ of dimension $\dim A > 1$ admit a noncritical holomorphic function $f: A \rightarrow \mathbb{C}$ of finite order?

Here we say that f is of finite order if $|f(z)| \leq \lambda \exp |z|^\mu$ holds for all $z \in A$ and for some pair of constants $\lambda, \mu > 0$.

Since such A is a Stein manifold, it admits a noncritical holomorphic function according to [F]. The construction in that paper is quite different from the one presented here even for Riemann surfaces, and it does not necessarily give a function of finite order when A is algebraic. The main

difficulty is that the closedness equation $d\omega = 0$ for a holomorphic 1-form, which is automatically satisfied on a Riemann surface, becomes a nontrivial condition when $\dim A > 1$. In particular, this condition is not preserved under multiplication by holomorphic functions, and hence one can not hope to adjust the periods in the same way as was done above.

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